APPROXIMATE ONE-DIMENSIONAL THERMAL CONDUCTION IN COMPOSITE CYLINDRICAL ELECTRONIC COMPONENTS

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Approximate one-dimensional formulas have been derived for the temperatures in composite rods, which incorporate the nonuniform temperature distribution in the transverse direction.

One frequently employs the one-dimensional approximation in calculating the temperature conditions in models for electronic equipment; this approximation gives satisfactory results for small values of the Biot criterion [1] for thermal models in the form of uniform rods of finite length. The accuracy of such models may be improved by refining the one-dimensional theory by estimating the transverse temperature variation. This is particularly important for multilayer rods made of different materials, since the exact analytical solutions that can generally be obtained only for simple boundary conditions are largely unsuitable because of the large computational volume.

We consider the heat transfer in a double cylinder heated at the base and in an atmosphere of temperature T_{am} in the case of heat transfer from the side of the cylinder to the environment by free convection and radiation, with a constant overall heat-transfer coefficient α; this amounts to solving Laplace's equation for the dimensionless temperature $u = (T - T_{am})/(T_0 - T_{am})$:

$$\Delta u(r, z) = 0 \tag{1}$$

subject to the boundary conditions

$$u(r, 0) = 1, \; u_z(r, l) = 0, \; -\lambda_2 u_r(a, z) = \alpha u(a, z).$$

The conditions for linking the solutions are equality of the temperatures and heat fluxes at the boundary between the outer and inner cylinders:

$$u(r_1 - 0, z) = u(r_1 + 0, z), \; k_\lambda u_r(r_1 - 0, z) = u_r(r_1 + 0, z). \tag{2}$$

In the particular case $k_\lambda = 1$ (uniform rod) the solution takes a particularly simple form as a series of eigenfunctions:

$$u(r, z) = 2\sum_{n=1}^{\infty} \frac{J_1(\mu_n) J_0\left(\mu_n \frac{r}{a}\right)}{\mu_n [J_0^2(\mu_n) + J_1^2(\mu_n)]} \cdot \frac{\operatorname{ch}\mu_n \frac{l-z}{a}}{\operatorname{ch}\mu_n \frac{l}{a}}, \tag{3}$$

where the μ_n are the positive roots of the equation

$$\mu J_1(\mu) = \mathrm{Bi}\, J_0(\mu). \tag{4}$$

It can be seen from the analysis of [1] that for reasonably long uniform rods ($l/a > 1.0$) the maximum deviations of the axial temperatures from the one-dimensional distribution are localized near the base of the rod, while the deviations decrease towards the top end, and the more so the greater the relative length of the rod. The physical treatment of the temperature distribution in such a model indicates that the one-dimensional solution is accompanied by some function that decreases rapidly as z increases, or a function of boundary-layer type in the terminology of [3]. For brevity, in what follows we restrict ourselves only to the case of a uniform rod, giving only the final formulas for a double rod.

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TABLE 1. Roots of the Characteristic Equation (9) for $k_a = 0.5$

k_λ	0,1	0,2	0,5	1,0	2,0	5,0	10,0
μ_1	4,6197	4,4700	4,1515	3,8317	3,4966	3,1326	2,9530
μ_2	6,5256	6,6251	6,8256	7,0156	7,2079	7,4159	7,5211
μ_3	10,8602	10,7228	10,4420	10,1735	9,9032	9,6174	9,4767
μ_4	12,7750	12,8873	13,1122	13,3237	13,5360	13,7634	13,8776
μ_5	17,1312	16,9979	16,7273	16,4706	16,2133	15,9410	15,8064
μ_6	19,0454	19,1622	19,3961	19,6159	19,8360	20,0712	20,1889

If Bi is small [2], the first positive root μ_1 in (4) can be taken as roughly equal to $\sqrt{2\mathrm{Bi}}$; one can refine this expression, i.e., the Taylor series in Bi, by incorporating terms dependent on higher powers of Bi:

$$\mu_1 = \sqrt{2\mathrm{Bi}}\left(1-\frac{\mathrm{Bi}}{8}\right)+0(\mathrm{Bi}^2). \tag{5}$$

The corresponding expression for root n of (4) is

$$\mu_n = \mu_n^{(0)} + \mathrm{Bi}/\mu_n^{(0)} + 0(\mathrm{Bi}^2),$$

where $\mu_n^{(0)}$ are the positive roots of $J_1(\mu) = 0$.

We expand the solution of (3) as a Taylor series near $\sqrt{2\mathrm{Bi}}$ for the first term in the sum and near $\mu_n^{(0)}$ for the subsequent terms and restrict ourselves to terms of the order of Bi to represent the solution of (3) in the form

$$u(r,\ z) = \frac{\mathrm{ch}\sqrt{2\mathrm{Bi}}\,\frac{l-z}{a}}{\mathrm{ch}\sqrt{2\mathrm{Bi}}\,\frac{l}{a}}\left(1-\frac{\mathrm{Bi}}{2}\cdot\frac{r^2}{a^2}\right)+\frac{\mathrm{Bi}\sqrt{2\mathrm{Bi}}\,\frac{l}{a}}{8\mathrm{ch}\sqrt{2\mathrm{Bi}}\,\frac{l}{a}}\left[\frac{\mathrm{ch}\sqrt{2\mathrm{Bi}}\,\frac{z}{a}}{\mathrm{ch}\sqrt{2\mathrm{Bi}}\,\frac{l}{a}}+\frac{z}{l}\,\mathrm{sh}\sqrt{2\mathrm{Bi}}\,\frac{l-z}{a}\right]+$$
$$+\frac{\mathrm{Bi}}{4}\,\frac{\mathrm{ch}\sqrt{2\mathrm{Bi}}\,\frac{l-z}{a}}{\mathrm{ch}\sqrt{2\mathrm{Bi}}\,\frac{l}{a}}+2\mathrm{Bi}\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n^{(0)}\frac{r}{a}\right)}{\mu_n^{(0)2}J_0(\mu_n^{(0)})}\exp\left[-\mu_n^{(0)}z/a\right]+0(\mathrm{Bi}^2); \tag{6}$$

on the assumption that the cylinder is long, i.e., that the ratio a/l is sufficiently small and $a/l = 0(\sqrt{\mathrm{Bi}})$.

Formula (6) allows one to calculate precisely the temperature distribution in a solid rod: the first term represents the one-dimensional solution, which is followed by corrections proportional to Bi, which incorporate the dependence of the temperature on the transverse coordinate, while the penultimate term is a function of boundary-layer type that decreases rapidly for z large. Then (6) allows one to calculate the temperature at any point in the cylinder to terms of magnitude Bi^2.

Analogous arguments for a double cylinder lead to the expression

$$u|_{r=0} = \frac{\mathrm{ch}\,m(1-z/l)}{\mathrm{ch}\,m}\left\{1+\mathrm{Bi}\,\frac{k_a^2(1-k_\lambda)\ln k_a+\frac{1}{4}[1+k_a^2(k_\lambda-1)(3k_a^2-2)]}{[1+k_a^2(k_\lambda-1)]^2}\right\}+$$
$$+\mathrm{Bi}\,\frac{\sqrt{2\mathrm{Bi}}\,\frac{l}{a}}{[1+k_a^2(k_\lambda-1)]^{5/2}}\cdot\frac{1}{\mathrm{ch}\,m}\left[\frac{1}{8}+\frac{1}{2}k_a^2(k_\lambda-1)-\frac{3}{8}k_a^4(k_\lambda-1)-\right.$$
$$\left.-\frac{k_a^4}{2}(k_\lambda-1)^2\ln k_a\right]\left[\frac{\mathrm{sh}\,m\,\frac{z}{l}}{\mathrm{ch}\,m}+\frac{z}{l}\,\mathrm{sh}\,m(1-z/l)\right]+\sum_{n=1}^{\infty}A_n\exp\left[-\mu_n^{(0)}z/a\right]+0(\mathrm{Bi}^2) \tag{7}$$

and

$$u|_{r=a} = \frac{\mathrm{ch}\,m(1-z/l)}{\mathrm{ch}\,m}\left\{1-\frac{\mathrm{Bi}}{2}\,\frac{1}{1+k_a^2(k_\lambda-1)}+\mathrm{Bi}\,\frac{k_a^2(k_\lambda-1)\ln k_a}{1+k_a^2(k_\lambda-1)}+\right.$$
$$\left.+\mathrm{Bi}\,\frac{k_a^2(1-k_\lambda)\ln k_a+\frac{1}{4}[1+k_a^2(k_\lambda-1)(3k_a^2-2)]}{[1+k_a^2(k_\lambda-1)]^2}\right\}+$$

TABLE 2. Thermal Amplitudes A_n and Values of ω_n for Some Values of the Biot Criterion

Bi	$k_\lambda = 0,1$		$k_\lambda = 1,0$	$k_\lambda = 10,0$	
	A_n	ω_n	A_n	A_n	ω_n
0,01	+1,0027 −0,0054 +0,0033	−0,0045 −0,9711 +0,5270	+1,0025 −0,0034 +0,0014	+1,0010 −0,0012 +0,0003	+0,0111 +1,1371 −0,5477
0,1	+1,0273 −0,0532 +0,0321	−0,0420 −0,9440 +0,5150	+1,0025 −0,0334 +0,0135	+1,0097 −0,0110 +0,0027	+0,1202 +1,0808 −0,5367
0,5	+1,1343 −0,2537 +0,1503	−0,1739 −0,8296 +0,4631	+1,1143 −0,1572 +0,0662	+1,0373 −0,0434 +0,0131	+0,6973 +0,8752 −0,4882

$$+\frac{\mathrm{Bi}\sqrt{2\mathrm{Bi}}\,\frac{l}{a}}{[1+k_a^2(k_\lambda-1)]^{5/2}}\cdot\frac{1}{\mathrm{ch}\, m}\left[\frac{1}{8}+\frac{1}{2}k_a^2(k_\lambda-1)-\frac{3}{8}k_a^4(k_\lambda-1)-\right.$$

$$\left.-\frac{k_a^4}{2}(k_\lambda-1)^2\ln k_a\right]\left[\frac{\mathrm{sh}\, m\frac{z}{l}}{\mathrm{ch}\, m}+\frac{z}{l}\,\mathrm{sh}\, m(1-z/l)\right]+\sum_{n=1}^{\infty}A_nR_n(a)\exp\left[-\mu_n^{(0)}z/a\right]+0\,(\mathrm{Bi}^2), \tag{8}$$

where $m = \frac{l}{a}\sqrt{\frac{2\mathrm{Bi}}{1+k_a^2(k_\lambda-1)}}$, $k_a = \frac{r_1}{a}$ and $k_\lambda = \lambda_1/\lambda_2$; here $R_n(a)$ is given by

$$R_n(a) = J_0(k_a\mu_n^{(0)})\,\frac{J_0(\mu_n^{(0)})-\omega_nY_0(\mu_n^{(0)})}{J_0(k_a\mu_n^{(0)})-\omega_nY_0(k_a\mu_n^{(0)})},$$

$$\text{where }\omega_n = \frac{\mathrm{Bi}J_0(\mu_n^{(0)})-\mu_n^{(0)}J_1(\mu_n^{(0)})}{\mathrm{Bi}Y_0(\mu_n^{(0)})-\mu_n^{(0)}Y_1(\mu_n^{(0)})},$$

and the $\mu_n^{(0)}$ are the positive roots of the characteristic equation

$$[k_\lambda J_1(k_a\mu)Y_0(k_a\mu)-J_0(k_a\mu)Y_1(k_a\mu)]\,J_1(\mu)-J_0(k_a\mu\,(J_1(k_a\mu)(k_\lambda-1)\,Y_1(\mu)=0, \tag{9}$$

and are given in Table 1. The amplitudes A_n are given by

$$A_n = \frac{[2\mathrm{Bi}/\mu_n^2]\,R_n(a)}{k_a^2(k_\lambda-1)\,[J_0^2(k_a\mu_n)-k_\lambda J_1^2(k_a\mu_n)]+(1+\mathrm{Bi}^2/\mu_n^2)R_n^2(a)}, \tag{10}$$

and were calculated for the value of Bi appearing in (7)-(8), and this is the value that has to be used in calculating the corresponding temperature. The results are given in Table 2. The roots μ_n are the roots of the characteristic equation analogous to (9) for Bi ≠ 0, which is obtained from the condition for ideal thermal contact between the internal and external cylinders in (2). If $k_\lambda = 1$ (or $k_a = 0$ and $k_a = 1$), formulas (7)-(8) become the corresponding expressions for a homogeneous cylinder.

Another important point is that (7)-(8) can be derived without resort to the exact solutions to (1)-(2) by transferring the methods of singular perturbations to the case of boundary conditions of the third kind [3].

If we restrict ourselves to the first term in the sum in the boundary-layer function in (7)-(8), one gets extremely accurate and reasonably simple formulas for the temperature in a long homogeneous or double rod. Calculations were performed with an M-220 computer using (7)-(8) with one term in the sum and gave very good agreement (with an error less than 1%) with the exact solution up to Bi of about 0.7-0.8, which are values commonly realized in practice with reasonably long rods and existing constructional materials.

NOTATION

- T is the absolute temperature;
- z is the coordinate along the rod;
- r is the radial coordinate;
- l is the linear dimension;

λ	is the thermal conductivity;
α	is the heat-transfer coefficient;
$J_0(x)$, $J_1(x)$, $Y_0(x)$, $Y_1(x)$	are the Bessel functions of the first and second kinds;
$Bi = \alpha a/\lambda_2$	is the Biot number;
$k_\lambda = \lambda_1/\lambda_2$; $k_a = r_1/a$	are the dimensionless coefficients.

Subscripts

am	is the ambient medium;
1	is the internal cylinder;
2	is the external cylinder;
n	is the series term.

LITERATURE CITED

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